



Normal hyperideals in Krasner (m, n) -hyperrings

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Abstract

Using a new definition, with respect to [21], for normal hyperideals in Krasner (m, n) -hyperrings, we show that the corresponding quotient structures are (m, n) -rings. We prove equivalency of (strongly) regular and (strongly) compatible relations on n -ary hypergroups. Also, the connection between (m, n) -hyperfields and maximal normal hyperideals is investigated.

1 Introduction

Algebraic hyperstructures, introduced by Marty [19] in 1934, represent an applied field of algebra, for instance in Euclidian and non Euclidian geometries, graphs and hypergraphs, binary relations, lattices, automata, cryptography, coding theory, artificial intelligence, probabilities, chemistry and so on (for more details, see [7] and also [6], [12], [25]).

N -ary hyperstructures, in particular n -ary hypergroups, were introduced by Davvaz and Vougiouklis [13] and represented a generalization of both hypergroups and n -ary groups, defined by Dornte [14] in 1928. Several authors have worked on this new topic of research, for instance (m, n) -hyperrings and (m, n) -hypermodules were introduced and studied in [1], [2], [3], [5], [8], [9], [11], [16], [17], [18], [20], [21], [22], [23] and [24].

On the other hand, one of the most important tools in algebraic hyperstructures is represented by strongly regular relations, in particular fundamental

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relations, which connect an algebraic hyperstructure to the associated algebraic structure.

In this paper, we propose and analyse a new definition for normal hyperideals in Krasner (m, n) -hyperrings, with respect to that one given in [21] and we show that these hyperideals correspond to strongly regular relations. We prove that the two definitions are equivalent. In this way, we show that (strongly) regular relations and (strongly) compatible relations on n -ary hypergroups are equivalent. Also, we investigate the connection between maximal normal hyperideals and (m, n) -hyperfields.

2 Preliminaries

A mapping $f : \underbrace{H \times \cdots \times H}_n \rightarrow \mathcal{P}^*(H)$ is called an n -ary hyperoperation,

where $\mathcal{P}^*(H)$ is the set of all nonempty subsets of H . An algebraic system (H, f) , where f is an n -ary hyperoperation defined on H , is called an n -ary hypergroupoid. We denote the sequence x_i, x_{i+1}, \dots, x_j by x_i^j , where for $j < i$, x_i^j is the empty set. Also, $f(x_1, \dots, x_i, y_{i+1}, \dots, y_j, z_{j+1}, \dots, z_n)$ is written as $f(x_1^i, y_{i+1}^j, z_{j+1}^n)$, and for $y_{i+1} = \cdots = y_j = y$, we write $f(x_1^i, y^{(j-i)}, z_{j+1}^n)$.

If f is an n -ary hyperoperation and $t = l(n - 1) + 1$, for some $l \geq 0$, then t -ary hyperoperation $f_{(l)}$ is given by

$$f_{(l)}\left(x_1^{l(n-1)+1}\right) = f\left(\underbrace{f(\dots, f(f(x_1^n), x_{n+1}^{2n-1}), \dots)}_l, x_{(l-1)(n-1)+1}^{l(n-1)+1}\right).$$

For nonempty subsets A_1, \dots, A_n of H we define $f(A_1^n) = \bigcup\{f(x_1^n) \mid x_i \in A_i, i = 1, \dots, n\}$. An n -ary hyperoperation f is called *associative* if

$$f\left(x_1^{i-1}, f(x_i^{n+i-1}), x_{n+i}^{2n-1}\right) = f\left(x_1^{j-1}, f(x_j^{n+j-1}), x_{n+j}^{2n-1}\right),$$

hold for all $1 \leq i < j \leq n$ and all $x_1, x_2, \dots, x_{2n-1} \in H$. An associative n -ary hypergroupoid is called an n -ary semihypergroup.

An n -ary semihypergroup (H, f) in which $H = f(a_1^{i-1}, H, a_{i+1}^n)$ for all $a_1^n \in H$ and $1 \leq i \leq n$, is called an n -ary hypergroup. Also, (H, f) is *commutative* if for all $\sigma \in \mathcal{S}_n$ and for all $a_1^n \in H$ we have $f(a_1^n) = f(a_{\sigma(1)}, \dots, a_{\sigma(n)})$.

Let (H, f) be an n -ary hypergroup and B be a nonempty subset of H . B is called an n -ary subhypergroup of H , if (B, f) is an n -ary hypergroup.

Definition 2.1. ([21]) Let (H, f) be a commutative n -ary hypergroup. (H, f) is called a canonical n -ary hypergroup, if

- (1) there exists a unique $e \in H$, such that for all $x \in H$, $f(x, e^{(n-1)}) = \{x\}$;

(2) for all $x \in H$ there exists a unique $x^{-1} \in H$, such that $e \in f(x, x^{-1}, e^{(n-2)})$;

(3) if $x \in f(x_1^n)$, then for all $1 \leq i \leq n$, we have

$$x_i \in f(x, x_1^{-1}, \dots, x_{i-1}^{-1}, x_{i+1}^{-1}, \dots, x_n^{-1}).$$

Definition 2.2. ([21]) An (m, n) -hyperring is an algebraic hyperstructure (R, f, g) which satisfies the following axioms:

(1) (R, f) is an m -ary hypergroup.

(2) (R, g) is an n -ary semihypergroup.

(3) The n -ary hyperoperation g is distributive with respect to the m -ary hyperoperation f , i.e., for all $a_1^{i-1}, a_{i+1}^n, x_1^m \in R$, and $1 \leq i \leq n$,

$$g(a_1^{i-1}, f(x_1^m), a_{i+1}^n) = f(g(a_1^{i-1}, x_1, a_{i+1}^n), \dots, g(a_1^{i-1}, x_m, a_{i+1}^n)).$$

An (m, n) -hyperring (R, f, g) is said to be Krasner if (R, f) is a canonical n -ary hypergroup and (R, g) is a n -ary semigroup such that 0 is a zero element (absorbing element) of the n -ary operation g , i.e. for all $x_2^n \in R$ we have

$$g(0, x_2^n) = g(x_2, 0, x_3^n) = \dots = g(x_2^n, 0).$$

Let I be a nonempty subset of Krasner (m, n) -hyperring (R, f, g) . We say that I is a *hyperideal* of R if (I, f) is a canonical m -ary subhypergroup of (R, f) and $g(x_1^{i-1}, I, x_{i+1}^n) \subseteq I$, for all $x_1^n \in R$ and $1 \leq i \leq n$.

Example 2.3. Consider the set of all integers, \mathbb{Z} , with $x \oplus y = \{x, y, x + y\}$ and $x \otimes y = \{x \cdot y\}$ for all $x, y \in \mathbb{Z}$, where “+” and “.” are ordinary addition and multiplication. For all $x, y, z \in \mathbb{Z}$, we have

$$(x \oplus y) \oplus z = \{x, y, z, x + y, x + z, y + z, x + y + z\} = x \oplus (y \oplus z),$$

and $x \oplus \mathbb{Z} = \bigcup_{y \in \mathbb{Z}} x \oplus y = \bigcup_{y \in \mathbb{Z}} \{x, y, x + y\} = \mathbb{Z}$. Hence (\mathbb{Z}, \oplus) is a hypergroup and also (\mathbb{Z}, \otimes) a semihypergroup, clearly. Moreover, we have

$$x \otimes (y \oplus z) = \{x \cdot y, x \cdot z, x \cdot y + x \cdot z\} = (x \otimes y) \oplus (x \otimes z),$$

and so $(\mathbb{Z}, \oplus, \otimes)$ is a hyperring. Note that although (\mathbb{Z}, \otimes) is a trivial semihypergroup but $(\mathbb{Z}, \oplus, \otimes)$ can not be seen as a Krasner hyperring. Now, we set $f(x_1^m) = \bigoplus_{i=1}^m x_i$ and $g(y_1^n) = \bigotimes_{i=1}^n y_i$, for $x_1^m, y_1^n \in \mathbb{Z}$. Then, (\mathbb{Z}, f, g) is an (m, n) -hyperring which can not be seen as a Krasner (m, n) -hyperring.

Example 2.4. ([21]) Suppose that (L, \vee, \wedge) is a relatively complemented distributive lattice and “ f ” and “ g ” are defined on L as follows:

$$f(a_1, a_2) = \{c \in L \mid a_1 \wedge c = a_2 \wedge c = a_1 \wedge a_2\}, \quad \forall a_1, a_2 \in L,$$

$$g(a_1^n) = \vee_{i=1}^n a_i, \quad \forall a_1^n \in L.$$

It follows that (L, f, g) is a Krasner $(2, n)$ -hyperring.

3 Normal hyperideals in Krasner (m, n) -hyperrings

In this section, we introduce normal hyperideals in Krasner (m, n) -hyperrings and investigate some results regarding them. Also, we compare our definition with the definition of normal hyperideals in [21]. Moreover, we analyse quotient Krasner (m, n) -hyperrings constructed by normal hyperideals. Finally, we investigate normal hyperideals in (m, n) -hyperfields and maximal hyperideals.

In what follows, let (R, f, g) be a Krasner (m, n) -hyperring.

Definition 3.1. A hyperideal I of (R, f, g) is said to be normal, if for all $r_1^m \in R$, we have $f\left(-r_1^{i-1}, f(r_1^{i-1}, I, r_{i+1}^m), -r_{i+1}^m\right) \subseteq I$.

Corollary 3.2. If I is a normal hyperideal of (R, f, g) , then for all $r_1^m \in R$, we have

$$f\left(-r_1^{i-1}, f(r_1^{i-1}, I, r_{i+1}^m), -r_{i+1}^m\right) = I.$$

Proof. Let I be a normal hyperideal. By the associativity of “ f ” and since (R, f) is canonical, it follows that

$$\begin{aligned} I &= f(I, 0^{(m-1)}) \\ &\subseteq f\left(I, f(-r_1, r_1, 0^{(m-2)}), \dots, f(-r_{i-1}, r_{i-1}, 0^{(m-2)}), \right. \\ &\quad \left. f(-r_{i+1}, r_{i+1}, 0^{(m-2)}), \dots, f(-r_m, r_m, 0^{(m-2)})\right) \\ &= f\left(-r_1^{i-1}, f(r_1^{i-1}, I, r_{i+1}^m), -r_{i+1}^m\right) \\ &\subseteq I. \end{aligned}$$

This completes the proof. □

According to [21], a hyperideal I of R is normal, if $f(-r, I, r, 0^{(m-3)}) \subseteq I$ for all $r \in R$. By the next theorem, we show that Definition 3.1 is equivalent to the definition of normal hyperideals defined by Mirvakili and Davvaz in [21].

Theorem 3.3. A hyperideal I of (R, f, g) is normal if and only if for every $r \in R$, we have

$$f(-r, I, r, 0^{(m-3)}) \subseteq I.$$

Proof. Let I be normal. By Definition 3.1 and since $0 \in R$, for all $r \in R$, we

have

$$\begin{aligned} f(-r, I, r, 0^{(m-3)}) &= f\left(f(-r, I, r, 0^{(m-3)}), 0^{(m-1)}\right) \\ &= f\left(-r, -0_2^{i-1}, f(r, 0_2^{i-1}, I, 0_{i+1}^m), -0_{i+1}^m\right) \\ &\subseteq I, \end{aligned}$$

where $-0_2^{i-1} = 0 = 0_{i+1}^m$. Now, let $f(-r_i, I, r_i, 0^{(m-3)}) \subseteq I$ for all $r_i \in R$ such that $1 \leq i \leq m$. For all I of R and $1 \leq i \leq m$, we have $I = f(I^{(i)}, 0^{(m-i)})$. It implies that

$$I = f(I, 0^{(m-1)}) \subseteq f\left(I, f(r, -r, 0^{(m-2)}), 0^{(m-2)}\right) = f(-r, I, r, 0^{(m-3)}).$$

Hence,

$$\begin{aligned} &f\left(-r_1^{i-1}, f(r_1^{i-1}, I, r_{i+1}^m), -r_{i+1}^m\right) \\ &= f\left(f(-r_1^{i-1}, f(r_1^{i-1}, I, r_{i+1}^m), -r_{i+1}^m), 0^{(m-1)}\right) \\ &= f\left(f(-r_1^{i-1}, 0, -r_{i+1}^m), I, f(r_1^{i-1}, 0, r_{i+1}^m), 0^{(m-3)}\right) \\ &\subseteq f\left(f(-r_1^{i-1}, 0, -r_{i+1}^m), f(-r_i, I, r_i, 0^{(m-3)}), f(r_1^{i-1}, 0, r_{i+1}^m), 0^{(m-3)}\right) \\ &= f\left(f(-r_1^m), f(I, 0^{(m-1)}), f(r_1^m), 0^{(m-3)}\right) \\ &\subseteq f\left(f(-r_1^m), f(I^{(m)}), f(r_1^m), f(0^{(m)})^{(m-3)}\right) \\ &= f\left(f(-r_1, I, r_1, 0^{(m-3)}), \dots, f(-r_m, I, r_m, 0^{(m-3)})\right) \\ &\subseteq I. \end{aligned}$$

Therefore, I is a normal hyperideal by Definition 3.1. □

In the following we state some lemmas which will be used in what follows.

Lemma 3.4. *Let (R, f, g) be a Krasner (m, n) -hyperring. Then for all $a_1^m, x, y \in R$ and $A, B, C \subseteq R$, we have*

- (1) $x \in f(B, a_2^m) \implies a_i \in f(x, -a_2^{i-1}, -B, -a_{i+1}^m), \forall 2 \leq i \leq m.$
- (2) $B \subseteq f(x, -a_2^m) \implies x \in f(B, a_2^m).$
- (3) $B \subseteq f(B, a_2^m) \implies a_i \in f(-a_2^{i-1}, B, -B, -a_{i+1}^m), \forall 2 \leq i \leq m.$
- (4) $f(x, -y, 0^{(m-2)}) \subseteq B \implies x \in f(y, B, 0^{(m-2)}).$

$$(5) f(A, B_2^m) \subseteq C \implies A \subseteq f(C, -B_2^m).$$

Proof. (1) Let $x \in f(B, a_2^m)$. There exists $2 \leq i \leq m$ such that

$$a_i \notin f(x, -a_2^{i-1}, -B, -a_{i+1}^m) = \bigcup_{t \in B} f(x, -a_2^{i-1}, -t, -a_{i+1}^m).$$

Then, for all $t \in B$ we have $a_i \notin f(x, -a_2^{i-1}, -t, -a_{i+1}^m)$. Hence, $x \notin f(a_2^{i-1}, a_i, t, a_{i+1}^m)$ for all $t \in B$. This implies that $x \notin f(B, a_2^m)$ which is a contradiction.

(2) Suppose that $x \notin f(B, a_2^m)$. Then for all $t \in B$ we have $x \notin f(t, a_2^m)$. Therefore, $t \notin f(x, -a_2^m)$, for all $t \in B$, which implies that $B \not\subseteq f(x, -a_2^m)$, that is a contradiction. By similar manner, (3) is proved.

(4) Since (R, f) is a canonical n -hypergroup and by hypothesis,

$$\begin{aligned} x \in f(x, 0^{(m-1)}) &\subseteq f\left(x, f(y, -y, 0^{(m-2)}), 0^{(m-2)}\right) \\ &= f\left(y, f(x, -y, 0^{(m-2)}), 0^{(m-2)}\right) \\ &\subseteq f(y, B, 0^{(m-2)}). \end{aligned}$$

(5) For all $x \in A$, we have

$$\begin{aligned} x &\in f(A, 0^{(m-1)}) \\ &\subseteq f\left(f(A, 0^{(m-1)}), f(B_2, -B_2, 0^{(m-2)}), \dots, f(B_m, -B_m, 0^{(m-2)})\right) \\ &= f\left(f(A, B_2^m), f(0, -B_2, 0^{(m-2)}), \dots, f(0, -B_m, 0^{(m-2)})\right) \\ &\subseteq f\left(f(A, B_2^m), -B_2^m\right) \\ &\subseteq f(C, -B_2^m). \end{aligned}$$

□

Lemma 3.5. *Let I be a normal hyperideal of (R, f, g) . Then for $x_{11}^{1m}, \dots, x_{m1}^{mm} \in R$, we have*

$$\begin{aligned} &f\left(f(x_{11}^{1(i-1)}, I, x_{1(i+1)}^{1m}), \dots, f(x_{m1}^{m(i-1)}, I, x_{m(i+1)}^{mm})\right) \\ &= f\left(f(x_{11}^{m1}), \dots, f(x_{1(i-1)}^{m(i-1)}), I, f(x_{1(i+1)}^{m(i+1)}), \dots, f(x_{1m}^{mm})\right). \end{aligned}$$

Proof. Let $t \in A = f\left(f(x_{11}^{1(i-1)}, I, x_{1(i+1)}^{1m}), \dots, f(x_{m1}^{m(i-1)}, I, x_{m(i+1)}^{mm})\right)$. Hence,

there exist $t_1^m \in I$ such that

$$\begin{aligned} t &\in f\left(f(x_{11}^{1(i-1)}, t_1, x_{1(i+1)}^{1m}), \dots, f(x_{m1}^{m(i-1)}, t_m, x_{m(i+1)}^{mm})\right) \\ &= f\left(f(x_{11}^{m1}), \dots, f(x_{1(i-1)}^{m(i-1)}), f(t_1^m), f(x_{1(i+1)}^{m(i+1)}), \dots, f(x_{1m}^{mm})\right) \\ &\subseteq f\left(f(x_{11}^{m1}), \dots, f(x_{1(i-1)}^{m(i-1)}), I, f(x_{1(i+1)}^{m(i+1)}), \dots, f(x_{1m}^{mm})\right) = B, \end{aligned}$$

hence $A \subseteq B$. Now, let $s \in f\left(f(x_{11}^{m1}), \dots, f(x_{1(i-1)}^{m(i-1)}), I, f(x_{1(i+1)}^{m(i+1)}), \dots, f(x_{1m}^{mm})\right)$. There exists $t \in I$ such that

$$\begin{aligned} s &\in f\left(f(x_{11}^{m1}), \dots, f(x_{1(i-1)}^{m(i-1)}), t, f(x_{1(i+1)}^{m(i+1)}), \dots, f(x_{1m}^{mm})\right) \\ &\subseteq f\left(f(x_{11}^{m1}), \dots, f(x_{1(i-1)}^{m(i-1)}), f(t, 0^{(m-1)}), f(x_{1(i+1)}^{m(i+1)}), \dots, f(x_{1m}^{mm})\right) \\ &= f\left(f(x_{11}^{1(i-1)}, t, x_{1(i+1)}^{1m}), f(x_{21}^{2(i-1)}, 0, x_{2(i+1)}^{2m}), \dots, f(x_{m1}^{m(i-1)}, 0, x_{m(i+1)}^{mm})\right) \\ &\subseteq f\left(f(x_{11}^{1(i-1)}, I, x_{1(i+1)}^{1m}), \dots, f(x_{m1}^{m(i-1)}, I, x_{m(i+1)}^{mm})\right). \end{aligned}$$

Therefore, $B \subseteq A$, and so the proof is completed. \square

Lemma 3.6. *Let I be a normal hyperideal of (R, f, g) .*

- (1) *For all $x, y \in R$, $y \in f(x, I, 0^{(m-2)})$ implies that $f(y, I, 0^{(m-2)}) = f(x, I, 0^{(m-2)})$.*
- (2) *If $y_i \in f(x_i, I, 0^{(m-2)})$ for $2 \leq i \leq m$ and $x_i, y_i \in R$, then $f(I, x_2^m) = f(I, y_2^m)$.*
- (3) *If $z, x_1^m \in R$ and $z \in f(x_1^m)$, then $f(z, I, 0^{(m-2)}) = f(f(x_1^m), I, 0^{(m-2)})$.*

Proof. The proof is similar to proof of Lemma 3.6 and 4.6 in [21]. \square

Now, we construct a quotient Krasner (m, n) -hyperring by a normal hyperideal and show that it is a trivial Krasner (m, n) -hyperring, that is a (m, n) -ring.

Let (R, f, g) be a Krasner (m, n) -hyperring and I be a hyperideal of R . Set

$$R/I = \{f(x_1^{i-1}, I, x_{i+1}^m) \mid x_1^{i-1}, x_{i+1}^m \in R, \quad 1 \leq i \leq m\}.$$

Define the m -ary hyperoperation “ F ” and n -ary operation “ G ” on R/I as follow:

$$\begin{aligned} F\left(f(x_{11}^{1(i-1)}, I, x_{1(i+1)}^{1m}), \dots, f(x_{m1}^{m(i-1)}, I, x_{m(i+1)}^{mm})\right) \\ = \left\{f(t_1^{i-1}, I, t_{i+1}^m) \mid t_i \in f(x_{1i}^{mi}); \quad 1 \leq i \leq m\right\} \end{aligned}$$

$$G\left(f(y_{11}^{1(i-1)}, I, y_{1(i+1)}^{1m}), \dots, f(y_{n1}^{n(i-1)}, I, y_{n(i+1)}^{nm})\right) = f\left(g(y_{11}^{n1}), \dots, g(y_{1(i-1)}^{n(i-1)}), I, g(y_{1(i+1)}^{n(i+1)}), \dots, g(y_{1m}^{nm})\right).$$

Then $(R/I, F, G)$ is a quotient Krasner (m, n) -hyperring constructed by a hyperideal I .

Now, let I be a normal hyperideal of (R, f, g) . Then, for $t_i \in f(x_{1i}^{mi})$ such that $1 \leq i \leq m$, we have

$$f(t_1^{i-1}, I, t_{i+1}^m) = f\left(f(x_{11}^{m1}), \dots, f(x_{1(i-1)}^{m(i-1)}), I, f(x_{1(i+1)}^{m(i+1)}), \dots, f(x_{1m}^{mm})\right),$$

by Lemma 3.6 (3). Moreover, by Lemma 3.5, it conclude that

$$f(t_1^{i-1}, I, t_{i+1}^m) = f\left(f(x_{11}^{1(i-1)}, I, x_{1(i+1)}^{1m}), \dots, f(x_{m1}^{m(i-1)}, I, x_{m(i+1)}^{mm})\right).$$

Hence, $F\left(f(x_{11}^{1(i-1)}, I, x_{1(i+1)}^{1m}), \dots, f(x_{m1}^{m(i-1)}, I, x_{m(i+1)}^{mm})\right)$ is the set of all $f(t_1^{i-1}, I, t_{i+1}^m)$ which satisfy in the following condition:

$$f(t_1^{i-1}, I, t_{i+1}^m) = f\left(f(x_{11}^{1(i-1)}, I, x_{1(i+1)}^{1m}), \dots, f(x_{m1}^{m(i-1)}, I, x_{m(i+1)}^{mm})\right).$$

Therefore, $(R/I, F, G)$, the quotient of a Krasner (m, n) -hyperring (R, f, g) constructed by a normal hyperideal I , defined by

$$F\left(f(x_{11}^{1(i-1)}, I, x_{1(i+1)}^{1m}), \dots, f(x_{m1}^{m(i-1)}, I, x_{m(i+1)}^{mm})\right) = \left\{ f\left(f(x_{11}^{1(i-1)}, I, x_{1(i+1)}^{1m}), \dots, f(x_{m1}^{m(i-1)}, I, x_{m(i+1)}^{mm})\right) \right\}$$

$$G\left(f(y_{11}^{1(i-1)}, I, y_{1(i+1)}^{1m}), \dots, f(y_{n1}^{n(i-1)}, I, y_{n(i+1)}^{nm})\right) = f\left(g(y_{11}^{n1}), \dots, g(y_{1(i-1)}^{n(i-1)}), I, g(y_{1(i+1)}^{n(i+1)}), \dots, g(y_{1m}^{nm})\right).$$

It means that m -ary and n -ary hyperoperations F and G are trivial.

Example 3.7. Suppose that $R = \{0, 1, 2, 3\}$ and define a 2-ary hyperoperation “+” and 4-ary operation “g” on R as follows:

$$g(x_1^4) = \begin{cases} 2, & \text{if } x_1^4 \in \{2, 3\} \\ 0, & \text{else} \end{cases} \quad \text{and} \quad \begin{array}{c|cccc} + & 0 & 1 & 2 & 3 \\ \hline 0 & \{0\} & \{1\} & \{2\} & \{3\} \\ 1 & \{1\} & \{0, 1\} & \{3\} & \{2, 3\} \\ 2 & \{2\} & \{3\} & \{0\} & \{1\} \\ 3 & \{3\} & \{2, 3\} & \{1\} & \{0, 1\} \end{array}$$

Then $(R, +, g)$ is a Krasner $(2, 4)$ -hyperring ([21]) and $I = \{0, 1\}$ is a normal hyperideal of R . Also, we have

$$\begin{aligned} 0 + I &= I = \{0, 1\} = \{1\} \cup \{0, 1\} = (1 + 0) \cup (1 + 1) = 1 + I, \\ 2 + I &= (2 + 0) \cup (2 + 1) = \{2\} \cup \{3\} = \{2, 3\} = 3 + I, \end{aligned}$$

hence, $R/I = \{0 + I, 2 + I\}$ and so

F	$0 + I$	$2 + I$
$0 + I$	$\{0 + I\}$	$\{2 + I\}$
$2 + I$	$\{2 + I\}$	$\{0 + I\}$

$$G(x_1 + I, \dots, x_4 + I) = g(x_1^4) + I = \begin{cases} 2 + I, & \text{if } x_1^4 \in \{2, 3\} \\ 0 + I, & \text{else} \end{cases}$$

hence $(R/I, F, G)$ is a Krasner $(2, 4)$ -hyperring.

The quotient of a Krasner (m, n) -hyperring, constructed by a normal hyperideal, is a trivial Krasner (m, n) -hyperring that is an (m, n) -ring. In the following, we show that (strongly) regular equivalence relations are equivalent with (strongly) compatible equivalence relations on n -ary semihypergroups. It helps us to show that normal hyperideals in Krasner (m, n) -hyperrings correspond to strongly regular relations on (m, n) -hyperrings (for more details about strongly regular relations see [6], [12] and [25]).

Let R be an equivalence relation on an n -ary semihypergroup (H, f) . For $A, B \in \mathcal{P}^*(H)$ we have

$$A\overline{R}B \iff \begin{cases} \forall a \in A, \exists b \in B; aRb \\ \forall b \in B, \exists a \in A; aRb \end{cases}$$

and $\overline{\overline{A\overline{R}B}}$ if and only if aRb for all $a \in A$ and $b \in B$.

Davvaz and Vougiouklis in [13] defined the concepts of compatible and strongly compatible relations on n -ary hypergroups as follow:

- (1) the equivalence relation R on (H, f) is called *compatible* if a_1Rb_1, \dots, a_nRb_n , then $f(a_1^n)\overline{R}f(b_1^n)$, for all $a_1^n, b_1^n \in H$.
- (2) R is said to be *strongly compatible* whenever a_iRb_i for all $1 \leq i \leq n$ implies that $f(a_1^n)\overline{\overline{R}}f(b_1^n)$.

Now, as a generalization of the concept of a (strongly) regular relation on semihypergroups ([6]), we say that an equivalence relation R on (H, f) is *i -regular* if xRy implies that $f(a_1^{i-1}, x, a_{i+1}^n)\overline{R}f(a_1^{i-1}, y, a_{i+1}^n)$ for all $x, y, a_1^n \in H$, and for $i \in \{1, \dots, n\}$. Also, if we have $f(a_1^{i-1}, x, a_{i+1}^n)\overline{\overline{R}}f(a_1^{i-1}, y, a_{i+1}^n)$,

then we say that R is *strongly i -regular*. An equivalence relation R is called (*strongly*) *regular* on (H, f) , if it is (strongly) i -regular for all $i \in \{1, \dots, n\}$.

In the following theorem, we show that (strongly) regular equivalence relations and (strongly) compatible equivalence relations are equivalent on n -ary semihypergroups.

Theorem 3.8. *Let (H, f) be an n -ary semihypergroup. An equivalence relation R on H is (strongly) compatible if and only if it is (strongly) regular on H .*

Proof. Let R be compatible and xRy for $x, y \in H$. Since R is equivalence relation, we have a_iRa_i for all $a_1^n \in H$. By compatibility of R , we have $f(a_1^{i-1}, x, a_{i+1}^n)\overline{R}f(a_1^{i-1}, y, a_{i+1}^n)$ for all $1 \leq i \leq n$. Thus, R is regular. Similarly, if R is strongly compatible, then R is strongly regular on H . Conversely, let R be a regular relation on H and a_iRb_i for all $1 \leq i \leq n$. Since R is regular, we have

$$\begin{aligned} a_1Rb_1 &\implies f(a_1, a_2^n)\overline{R}f(b_1, a_2^n) \\ a_2Rb_2 &\implies f(b_1, a_2, a_3^n)\overline{R}f(b_1, b_2, a_3^n) \\ a_3Rb_3 &\implies f(b_1, b_2, a_3, a_4^n)\overline{R}f(b_1, b_2, b_3, a_4^n) \\ &\vdots \\ a_nRb_n &\implies f(b_1^{n-1}, a_n)\overline{R}f(b_1^{n-1}, b_n). \end{aligned}$$

Since R is an equivalence relation, it follows that $f(a_1^n)\overline{R}f(b_1^n)$ which implies that R is a compatible relation. Now, let R be strongly regular and a_iRb_i for all $1 \leq i \leq n$. Also, let $x \in f(a_1^n)$ and $y \in f(b_1^n)$. Since $f(b_1, a_2^n), f(b_1, b_2, a_3^n), f(b_1, b_2, b_3, a_4^n), \dots, f(b_1^{n-1}, a_n)$ are non-empty subsets of H , there exist $t_1 \in f(b_1, a_2^n), t_2 \in f(b_1, b_2, a_3^n), t_3 \in f(b_1, b_2, b_3, a_4^n), \dots, t_{n-1} \in f(b_1^{n-1}, a_n)$. Since a_1Rb_1, \dots, a_nRb_n and R is strongly regular, we have $xRt_1, t_1Rt_2, \dots, t_{n-1}Ry$. This implies that xRy and so $f(a_1^n)\overline{R}f(b_1^n)$. Then, R is a strongly compatible relation on H . \square

Now, consider the following relation on a Krasner (m, n) -hyperring (R, f, g) by normal hyperideal I of R :

$$xI^*y \text{ if and only if } f(x, -y, 0^{(m-2)}) \cap I \neq \emptyset, \quad \forall x, y \in R.$$

Mirvakili and Davvaz in [21] showed that I^* is an equivalence relation on R and if $I^*[x]$ is equivalence class of $x \in R$, then $I^*[x] = f(x, I, 0^{(m-2)})$. In the following, we show that I^* is a strongly regular (compatible) relation on R .

Theorem 3.9. *Let (R, f, g) be a Krasner (m, n) -hyperring and I a normal hyperideal of R . Then, I^* is a strongly regular relation on (R, f) .*

Proof. Let xI^*y for $x, y \in R$. We must show that $f(x, a_2^m)\overline{I^*}f(y, a_2^m)$ for all $a_2^m \in R$. Hence, let $a_2^m \in R$, $t \in f(x, a_2^m)$ and $u \in f(y, a_2^m)$. Then, by Lemma 3.5 and Lemma 3.6, we have

$$\begin{aligned} f(t, I, 0^{(m-2)}) &= f\left(f(x, a_2^m), I, 0^{(m-2)}\right) \\ &= f\left(f(x, I, 0^{(m-2)}), f(a_2, I, 0^{(m-2)}), \dots, f(a_m, I, 0^{(m-2)})\right) \\ &= f\left(f(y, I, 0^{(m-2)}), f(a_2, I, 0^{(m-2)}), \dots, f(a_m, I, 0^{(m-2)})\right) \\ &= f\left(f(y, a_2^m), I, 0^{(m-2)}\right) \\ &= f(u, I, 0^{(m-2)}). \end{aligned}$$

It follows that $I^*[t] = I^*[u]$ and so tI^*u . Therefore, I^* is a strongly regular relation. \square

Corollary 3.10. *I^* is a strongly compatible relation on (R, f) .*

Proof. It is straightforward by Theorem 3.8 and Theorem 3.9. \square

According to [4], if (H, f) is an n -ary hypergroup and R is a strongly compatible equivalence relation on H , then the quotient $H/R = \{R(x) \mid x \in H\}$ endowed with n -ary operation (trivial n -ary hyperoperation) f/R is an n -ary group (trivial n -ary hypergroup), where for all $R(x_1), \dots, R(x_n)$ of R/H

$$f/R\left(R(x_1), \dots, R(x_n)\right) = \left\{R(z)\right\}, \quad \forall z \in f(x_1^n).$$

Hence, we can conclude the following corollary:

Corollary 3.11. *If (R, f, g) is a Krasner (m, n) -hyperring and I is a normal hyperideal of R , then $(R/I^*, f/I^*, g/I^*)$ is an (m, n) -ring (trivial Krasner (m, n) -hyperring), where $g/I^*(I^*[y_1], \dots, I^*[y_n]) = I^*[g(y_1^n)]$ for all $I^*[y_1], \dots, I^*[y_n] \in R/I^*$.*

Now, if we put $R/I = \{f(x, I, 0^{(m-2)}) \mid x \in R\}$ such that I is a normal hyperideal of the Krasner (m, n) -hyperring (R, f, g) , then we have $R/I = \{I^*[x] \mid x \in R\} = R/I^*$, and

$$\begin{aligned} F\left(f(x_1, I, 0^{(m-2)}), \dots, f(x_m, I, 0^{(m-2)})\right) &= \{f(t, I, 0^{(m-2)}) \mid t \in f(x_1^m)\} \\ &= \{I^*[t] \mid t \in f(x_1^m)\} \\ &= f/I^*(I^*[x_1], \dots, I^*[x_m]), \end{aligned}$$

and similarly

$$\begin{aligned} G\left(f(y_1, I, 0^{(m-2)}), \dots, f(y_n, I, 0^{(m-2)})\right) &= f\left(g(y_1^n), I, 0^{(m-2)}\right) \\ &= I^*[g(y_1^n)] \\ &= g/I^*(I^*[y_1], \dots, I^*[y_n]). \end{aligned}$$

Therefore, $(R/I, F, G) = (R/I^*, f/I^*, g/I^*)$. This implies that normal hyperideals in Krasner (m, n) -hyperrings correspond to strongly regular relations on (m, n) -hyperrings and the quotient of a Krasner (m, n) -hyperring, constructed by a normal hyperideal, is an (m, n) -ring.

In what follows, we investigate the connection between maximal normal hyperideals and (m, n) -hyperfields. Let (R, f, g) be a commutative Krasner (m, n) -hyperring with a scalar identity 1_R , that is $x = g(x, 1_R^{(n-1)})$ for all $x \in R$. An element $u \in R$ is said to be *invertible*, if there exists an element $b \in R$ such that $1_R = g(u, b, 1_R^{(n-2)})$. We say (R, f, g) is an (m, n) -hyperfield, if any nonzero element of R is invertible.

Theorem 3.12. *Let (R, f, g) be a commutative Krasner (m, n) -hyperring with a scalar identity and I be a normal hyperideal of R . Then, I is maximal if and only if R/I is a trivial (m, n) -hyperfield (that is an (m, n) -field).*

Proof. Let I be a maximal hyperideal of R . Since R is commutative and has a scalar identity, then $(R/I, F, G)$ is also commutative and has a scalar identity. Suppose that $f(a, I, 0^{(m-2)}) \in R/I$ such that $f(a, I, 0^{(m-2)}) \neq I$. Then, by Lemma 3.4 in [21], it follows that $a \notin I$. Hence, $R = \langle I, a \rangle$ and so there exist $m \in I$ and $r \in R$ such that $1_R \in f(m, g(r, a, 1_R^{(n-2)}), 0^{(m-2)})$. Thus, by Lemma 3.6,

$$\begin{aligned} f(1_R, I, 0^{(m-2)}) &= f\left(f(m, g(r, a, 1_R^{(n-2)}), 0^{(m-2)}), I, 0^{(m-2)}\right) \\ &= f\left(g(r, a, 1_R^{(n-2)}), f(m, I, 0^{(m-2)}), 0^{(m-2)}\right) \\ &= f(g(r, a, 1_R^{(n-2)}), I, 0^{(m-2)}) \\ &= G(f(r, I, 0^{(m-2)}), f(a, I, 0^{(m-2)}), f(1_R, I, 0^{(m-2)})^{(n-2)}). \end{aligned}$$

Hence, $f(a, I, 0^{(m-2)})$ has an inverse. Therefore, all nonzero element of R/I are invertible and so R/I is an (m, n) -hyperfield, that is an (m, n) -field. Conversely, let R/I be an (m, n) -hyperfield, hence $I \neq R$. There exists a hyperideal L of R such that $I \subset L \subseteq R$. Hence, there exists $a \in L$ such that $a \notin I$, whence $f(a, I, 0^{(m-2)}) \neq I$. Since R/I is an (m, n) -hyperfield, it follows

that $I \neq f(r, I, 0^{(m-2)}) \in R/I$ and

$$\begin{aligned} f(1_R, I, 0^{(m-2)}) &= G(f(r, I, 0^{(m-2)}), f(a, I, 0^{(m-2)}), f(1_R, I, 0^{(m-2)})^{(n-2)}) \\ &= f(g(r, a, 1_R^{(n-2)}), I, 0^{(m-2)}). \end{aligned}$$

Hence, since (R, f) is canonical and L is a hyperideal, it follows that

$$\begin{aligned} 1_R \in f(1_R, 0^{(m-1)}) &\subseteq f(1_R, I, 0^{(m-2)}) \\ &= f(g(r, a, 1_R^{(n-2)}), I, 0^{(m-2)}) \\ &\subseteq f(L, L, 0^{(m-2)}) \\ &\subseteq L. \end{aligned}$$

Therefore, $L = R$ and so I is a maximal hyperideal of R . □

4 Conclusion

We give a new definition for a normal hyperideal in a Krasner (m, n) -hyperring and we show that the corresponding quotient Krasner (m, n) -hyperring is an (m, n) -ring, which means that normal hyperideals act in a similar way as strongly regular relations in (m, n) -hyperrings.

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